On nonlocal perturbations of integral kernels

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Abstract

We give sufficient conditions for a nonlocal perturbation of an integral kernel to be locally in time comparable with the kernel.

1 Introduction and Preliminaries

We may add jumps to a Markov process by adding a nonlocal operator to its generator. We will be concerned with estimates of the resulting transition kernels. In fact, we will give measure-theoretic conditions on perturbations of rather general integral kernels which guarantee comparability of the resulting perturbation series with the original kernel. We are motivated by recent estimates of local or Schrödinger perturbations of integral kernels in [5], and nonlocal perturbations of the Green functions in [8] and [9].

We will perturb the so-called forward kernels. The perturbing kernels will be nonlocal in space, but instantaneous in time. The resulting perturbation and the original kernel turn out to be comparable locally in time and globally in space under an appropriate integral condition on the first term of the perturbation series. The estimates allow to control e.g. the evolution of the distribution of a population affected by dislocations and creation of population members. Transition kernels of Markov processes are our main

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motivation for this work, however we do not generally impose Chapman-Kolmogorov condition on the kernels.

The paper is composed as follows. In Section 2 we formulate our main estimates: Theorem 2.2 for kernels and Theorem 2.3 and 2.4 for kernel densities. In Section 3 we note that nonlocal perturbations of transition kernels are transition kernels, too. In Section 4 we briefly mention signed perturbations and give lower bounds for negative perturbations of transition kernels. In Section 5 we show the extra work that needs to be done in order to apply our results in specific situations (we focus on nonlocal perturbations of the transition density of the fractional Laplacian), we describe the perturbation series in terms of generators and fundamental solutions and we illustrate the effect which nonlocal perturbations have on jump intensity of Markov processes.

We note that Theorems 2.2, 2.3 and 2.4 generalize the main estimates of [5] for Schrödinger perturbations of integral kernels. The reader may find in [5] and a related paper [3] some general comments on this research program, and more applications, e.g. to Weyl fractional integrals ([5, Example 3]).

2 Main results

We first recall, after [7], some properties of kernels. Let $(E, \mathcal{E})$ be a measurable space. A kernel on E is a map K from $E \times \mathcal{E}$ to $[0, \infty]$ such that

$x \mapsto K(x, A)$ is $\mathcal{E}$-measurable for all $A \in \mathcal{E}$, and

$A \mapsto K(x, A)$ is countably additive for all $x \in E$.

Consider kernels K and J on E. The map

$$(x, A) \mapsto \int_E K(x, dy) J(y, A)$$

from $(E \times \mathcal{E})$ to $[0, \infty]$ is another kernel on E, called the *composition* of K and J, and denoted KJ. We let $K_n = K_{n-1}JK(s, x, A) = (KJ)^n K$, $n = 0, 1, \ldots$. The composition of kernels is associative, which yields the following lemma.

Lemma 2.1. *$K_n = K_{n-1-m}JK_m$ for all $n \in \mathbb{N}$ and $m = 0, 1, \ldots, n-1$.*

We define the *perturbation*, $\widetilde{K}$, of K by J, via the *perturbation series*,

$$\widetilde{K} = \sum_{n=0}^{\infty} K_n = \sum_{n=0}^{\infty} (KJ)^n K. \tag{1}$$

Of course, $K \le \widetilde{K}$, and the following *perturbation formula* holds,

$$\widetilde{K} = K + \widetilde{K}JK. \tag{2}$$

Below we will prove upper bounds for $\widetilde{K}$ under additional conditions on K, J and $K_1 = KJK$.

Consider a set X (the state space) with σ-algebra $\mathcal{M}$, the real line $\mathbb{R}$ (the time) equipped with the Borel sets $\mathcal{B}_{\mathbb{R}}$, and consider the space-time

$$E := \mathbb{R} \times X,$$

with the product σ-algebra $\mathcal{E} = \mathcal{B}_{\mathbb{R}} \times \mathcal{M}$. Let $\eta \in [0, \infty)$ and a function $Q : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ satisfy the following condition of *super-additivity*:

$$Q(u, r) + Q(r, v) \le Q(u, v) \quad \text{for all } u < r < v.$$

Let K be a kernel on E. We will assume that K is a *forward* kernel, i.e.

$$K(s, x, A) = 0 \qquad \text{whenever } A \subseteq (-\infty, s] \times X \quad (A \in \mathcal{E},\ s \in \mathbb{R}).$$

Let $\delta_s(B) = \mathbb{1}_B(s)$ denote the Dirac measure at $s \in \mathbb{R}$. Assume that kernel J on $(E, \mathcal{E})$ is instantaneous in time, i.e. $J(s, x, dtdy) = J(s, x, dtdy)\mathbb{1}_{t=s}$ or $J(s, x, dtdy) = j(s, x, dy)\delta_s(dt)$, where $j(s, x, dy) = J(s, x, \mathbb{R} \times dy)$. Assume

$$KJK(s, x, A) \le \int_A [\eta + Q(s, t)]K(s, x, dtdy). \tag{3}$$

Theorem 2.2. *Under the assumption, for all $n = 1, 2, \ldots$, and $(s, x) \in E$,*

$$K_n(s, x, dtdy) \le K_{n-1}(s, x, dtdy)\left[\eta + \frac{Q(s,t)}{n}\right] \tag{4}$$

$$\le K(s, x, dtdy)\prod_{k=1}^{n}\left[\eta + \frac{Q(s,t)}{k}\right]. \tag{5}$$

If $0 < \eta < 1$, then for all $(s, x) \in E$,

$$\widetilde{K}(s, x, dtdy) \le K(s, x, dtdy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}. \tag{6}$$

If $\eta = 0$, then for all $(s, x) \in E$,

$$\widetilde{K}(s, x, dtdy) \le K(s, x, dtdy)e^{Q(s,t)}. \tag{7}$$

Proof. (3) yields (4) for $n = 1$. By induction,

$$
\begin{aligned}
&(n+1)K_{n+1}(s,x,A) = nK_nJK(s,x,A) + K_{n-1}JK_1(s,x,A) \\
= \;& n\int\limits_E K_n(s,x,dudz)(JK)(u,z,A) + \int\limits_E (K_{n-1}J)(s,x,dudz)K_1(u,z,A) \\
\le \;& n\int\limits_E \left(\eta + \frac{Q(s,u)}{n}\right) K_{n-1}(s,x,dudz)(JK)(u,z,A) \\
& + \int\limits_E (K_{n-1}J)(s,x,dudz) \int\limits_A (\eta + Q(u,t))K(u,z,dtdy) \\
= \;& (n+1)\eta K_n(s,x,A) \\
& + \int\limits_E Q(s,u)K_{n-1}(s,x,dudz) \int\limits_E j(u,z,dz_1)\delta_u(du_1)K(u_1,z_1,A) \\
& + \int\limits_E \int\limits_E K_{n-1}(s,x,du_1dz_1)j(u_1,z_1,dz)\delta_{u_1}(du) \int\limits_A Q(u,t)K(u,z,dtdy) \\
= \;& (n+1)\eta K_n(s,x,A) + \int\limits_E Q(s,u)K_{n-1}(s,x,dudz) \int\limits_X j(u,z,dz_1)K(u,z_1,A) \\
& + \int\limits_X \int\limits_E K_{n-1}(s,x,du_1dz_1)j(u_1,z_1,dz) \int\limits_A Q(u_1,t)K(u_1,z,dtdy) \\
\le \;& (n+1)\eta K_n(s,x,A) \\
& + \int\limits_A Q(s,t) \int\limits_E K_{n-1}(s,x,dudz) \int\limits_E j(u,z,dz_1)\delta_u(du_1)K(u_1,z_1,dtdy) \\
= \;& (n+1)\eta K_n(s,x,A) + \int\limits_A Q(s,t) \int\limits_E K_{n-1}(s,x,dudz)(JK)(u,z,dtdy) \\
= \;& (n+1)\int\limits_A \left(\eta + \frac{Q(s,t)}{n+1}\right) K_n(s,x,dtdy).
\end{aligned}
$$

(5) follows from (4), (7) results from Taylor's expansion of the exponential function, and (6) follows from the Taylor series

$$
(1-\eta)^{-a} = \sum_{n=0}^{\infty} \frac{\eta^n (a)_n}{n!},
$$

where $0 < \eta < 1$, $a \in \mathbb{R}$, and $(a)_n = a(a+1)\cdots(a+n-1)$. $\square$

Theorem 2.2 has two *fine* or *pointwise* variants, which we will state under suitable conditions. Fix a (nonnegative) σ-finite, non-atomic measure

$$dt = \mu(dt)$$

on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and a function $k(s, x, t, A)$ defined for $s < t$, $x \in X$, $A \in \mathcal{M}$, such that $k(s, x, t, dy)dt$ is a forward kernel and $(s, x) \mapsto k(s, x, t, A)$ is jointly measurable for all $t \in \mathbb{R}$ and $A \in \mathcal{M}$. Let $k_0 = k$, and for $n = 1, 2, \ldots$,

$$k_n(s, x, t, A) = \int_s^t \int_X k_{n-1}(s, x, u, dz) \int_X j(u, z, dw) k(u, w, t, A) du.$$

The perturbation, $\widetilde{k}$, of k by J, is defined as

$$\widetilde{k} = \sum_{n=0}^{\infty} k_n.$$

Assume that

$$\int_s^t \int_X k(s, x, u, dz) \int_X j(u, z, dw) k(u, w, t, A) du \le [\eta + Q(s, t)] k(s, x, t, A).$$

Theorem 2.3. *Under the assumptions, for all $n = 1, 2, \ldots,$ and $(s, x) \in E$,*

$$\begin{aligned} k_n(s, x, t, dy) &\le k_{n-1}(s, x, t, dy) \left[\eta + \frac{Q(s, t)}{n}\right] \\ &\le k(s, x, t, dy) \prod_{l=1}^{n} \left[\eta + \frac{Q(s, t)}{l}\right]. \end{aligned}$$

If $0 < \eta < 1$, then for all $(s, x) \in E$,

$$\widetilde{k}(s, x, t, dy) \le k(s, x, t, dy) \left(\frac{1}{1 - \eta}\right)^{1 + Q(s,t)/\eta}.$$

If $\eta = 0$, then for all $(s, x) \in E$,

$$\widetilde{k}(s, x, t, dy) \le k(s, x, t, dy) e^{Q(s,t)}.$$

We skip the proof, because it is similar to the proof of Theorem 2.2. For the *finest* variant of Theorem 2.2, we fix a σ-finite measure

$$dz = m(dz)$$

on $(X, \mathcal{M})$. We consider function $\kappa(s,x,t,y) \geq 0$, $s,t \in \mathbb{R}$, $x,y \in X$, such that $\kappa(s,x,t,y)dtdy$ is a forward kernel and $(s,x) \mapsto k(s,x,t,y)$ is jointly measurable for all $t \in \mathbb{R}$ and $y \in X$. We call such κ a (forward) *kernel density* (see [5]). We define $\kappa_0(s,x,t,y) = \kappa(s,x,t,y)$, and

$$\kappa_n(s,x,t,y) = \int_s^t \int_X \kappa_{n-1}(s,x,u,z) \int_X j(u,z,dw)\kappa(u,w,t,y)\, dz\, du\,,$$

where $n = 1, 2, \ldots$. We assume that for all $s < t \in \mathbb{R}$ and $x, y \in X$,

$$\int_s^t \int_X \kappa(s,x,u,z) \int_X j(u,z,dw)\kappa(u,w,t,y) dz du \leq [\eta + Q(s,t)]\kappa(s,x,t,y).$$

Theorem 2.4. *Under the assumptions, for $n = 1, 2, \ldots$, $s < t$ and $x, y \in X$,*

$$\begin{aligned}\kappa_n(s,x,t,y) &\leq \kappa_{n-1}(s,x,t,y)\left[\eta + \frac{Q(s,t)}{n}\right] \\ &\leq \kappa(s,x,t,y)\prod_{k=1}^{n}\left[\eta + \frac{Q(s,t)}{k}\right].\end{aligned}$$

If $0 < \eta < 1$, then for all $s < t$ and $x, y \in X$,

$$\widetilde{\kappa}(s,x,t,y) \leq \kappa(s,x,t,y)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}.$$

If $\eta = 0$, then for all $s < t$ and $x, y \in X$,

$$\widetilde{\kappa}(s,x,t,y) \leq \kappa(s,x,t,y)e^{Q(s,t)}.$$

We skip the proof, because it is similar to the proof of Theorem 2.2.

3 Transition kernels

Let k above (note the joint measurability) be a *transition kernel* i.e. additionally satisfy the Chapman-Kolmogorov conditions for $s < u < t$, $A \in \mathcal{M}$,

$$\int_X k(s,x,u,dz)k(u,z,t,A) = k(s,x,t,A).$$

Following [2], we will show that $\widetilde{k}$ is a transition kernel, too.

Lemma 3.1. *For all $s < u < t$, $x, y \in X$, $A \in \mathcal{M}$ and $n = 0, 1, \ldots$,*

$$\sum_{m=0}^{n} \int_X k_m(s, x, u, dz) k_{n-m}(u, z, t, A) = k_n(s, x, t, A) \tag{8}$$

Proof. We note that (8) is true for $n = 0$ by fact that k is a transition kernel and satisfies the Chapman-Kolmogorov equation. Assume that $n \geq 1$ and (8) holds for $n - 1$. The sum of the first n terms in (8) can be dealt with by induction:

$$\begin{aligned}
&\sum_{m=0}^{n-1} \int_X k_m(s, x, u, dz) k_{n-m}(u, z, t, A) \qquad (9)\\
=\ &\sum_{m=0}^{n-1} \int_X k_m(s, x, u, dz) \int_u^t \int_X k_{n-m-1}(u, z, r, dw)\\
&\times \int_X j(r, w, dw_1) k(r, w_1, t, A) dr\\
=\ &\int_u^t \int_X \int_X j(r, w, dw_1) k(r, w_1, t, A)\\
&\times \sum_{m=0}^{n-1} \int_X k_m(s, x, u, dz) k_{(n-1)-m}(u, z, r, dw) dr\\
=\ &\int_u^t \int_X k_{n-1}(s, x, r, dw) \int_X j(r, w, dw_1) k(r, w_1, t, A) dr.
\end{aligned}$$

The $(n+1)$-st term is

$$\begin{aligned}
&\int_X k_n(s, x, u, dz) k_0(u, z, t, A) \qquad (10)\\
=\ &\int_X \int_s^u \int_X k_{n-1}(s, x, r, dw) \int_X j(r, w, dw_1) k(r, w_1, u, dz) dr\ k(u, z, t, A)\\
=\ &\int_s^u \int_X k_{n-1}(s, x, r, dw) \int_X j(r, w, w_1) k(r, w_1, t, A) dr,
\end{aligned}$$

and (8) follows on adding (9) and (10). $\square$

Lemma 3.2. *For all $s < u < t$, $x, y \in \mathbb{R}^d$ and $A \in \mathcal{M}$,*

$$\int_X \widetilde{k}(s,x,u,dz)\widetilde{k}(u,z,t,A) = \widetilde{k}(s,x,t,A).$$

We refer to [2, Lemma 2] for the proof. Thus, $\widetilde{k}$ is a transition kernel.

Similarly, the above considered function κ (note the joint measurability) is called transition density if it satisfies Chapman-Kolmogorov equations pointwise. In an analogous way we then prove that $\widetilde{\kappa}$ defined above is a transition density, if so is κ.

4 Signed perturbation

The following discussion is modeled after [2]. We will consider perturbation of K by $j(s,x,dy)m(s,x,y)$, where $m : \mathbb{R} \times X \times X \to [-1,1]$ is jointly measurable. If $\widetilde{K}$ is finite, then the perturbation series resulting from jm is absolutely convergent, and the perturbation formula extends to this case. For instance, the perturbation of K by $-J$ is

$$\widetilde{K}^- = \sum_{n=0}^{\infty}(-1)^n(KJ)^nK,$$

and

$$\widetilde{K}^- = K - \widetilde{K}^-JK.$$

Under the assumptions of Theorem 2.2 we get

$$\begin{aligned}\widetilde{K}^- &= [K - KJK] + [(KJ)^2K - (KJ)^3K] - \dots \\ &\geq \sum_{n=0,2,\dots}\left(1 - \eta - \frac{Q(s,t)}{n+1}\right)(KJ)^nK \geq \frac{1-\eta}{2}K,\end{aligned}$$

if $Q(s,t) \leq (1-\eta)/2$, and

$$\begin{aligned}\widetilde{K}^- &= K - [KJK - (KJ)^2K] - [(KJ)^3K - (KJ)^4K] - \dots \\ &\leq K - \sum_{n=1,3,\dots}\left(1 - \eta - \frac{Q(s,t)}{n+1}\right)(KJ)^nK \leq K, \qquad (11)\end{aligned}$$

if $Q(s,t) \leq 2(1-\eta)$.

If k is a *transition* kernel, $s = u_0 < u_1 < \ldots < u_{n-1} < u_n = t$ and $Q(u_{l-1}, u_l) \leq (1-\eta)/2$ for $l = 1, 2, \ldots, n$, then

$$\begin{aligned}
\widetilde{k}(s,x,t,A) &= \int\limits_X \ldots \int\limits_X \widetilde{k}(s,x,u_1,dz_1)\widetilde{k}(u_1,z_1,u_2,dz_2)\ldots\widetilde{k}(u_{n-1},z_{n-1},t,A) \\
&\geq \left(\frac{1-\eta}{2}\right)^n \int\limits_X \ldots \int\limits_X k(s,x,u_1,dz_1)k(u_1,z_1,u_2,dz_2)\ldots k(u_{n-1},z_{n-1},t,A) \\
&= \left(\frac{1-\eta}{2}\right)^n k(s,x,t,A). \qquad (12)
\end{aligned}$$

If $Q(s,t) \leq h(t-s)$ and $h(0^+) = 0$, then global lower bounds for $\widetilde{k}^-$ easily follow, and so this upper bound obtains from the perturbation formula:

$$\widetilde{k}^- \leq k.$$

Analogous results hold pointwise for *transition* densities κ.

We remark that our estimates of transition kernels may give estimates of the corresponding resolvent and potential operators provided we also have bounds for large times (see [4, Lemma 7] and (21) in this connection).

5 Applications

Verification of (3) usually requires some work. Here is a case study. Let $\alpha \in (0,2)$. Consider the convolution semigroup of functions defined as

$$p_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ixu} e^{-t|u|^\alpha} du \quad \text{for} \quad t > 0\ x \in \mathbb{R}^d. \qquad (13)$$

The semigroup is generated by the fractional Laplacian $\Delta^{\alpha/2}$ ([1]). By (13),

$$p_t(x) = t^{-\frac{d}{\alpha}} p_1(t^{-\frac{1}{\alpha}} x).$$

By subordination ([1]) we see that $p_t(x)$ is decreasing in $|x|$:

$$p_t(x) \geq p_t(y) \quad \text{if} \quad |x| \leq |y|. \qquad (14)$$

We write $f(a,\ldots,z) \approx g(a,\ldots,z)$ if there is a number $0 < C < \infty$ independent of $a,\ldots,z$, i.e. a *constant*, such that $C^{-1}f(a,\ldots,z) \leq g(a,\ldots,z) \leq Cf(a,\ldots,z)$ for all $a,\ldots,z$. We have (see, e.g., [4]),

$$p_t(x) \approx t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}}. \qquad (15)$$

Noteworthy, $t^{-\frac{d}{\alpha}} \le t/|x|^{d+\alpha}$ iff $t \le |x|^\alpha$. We observe the following property:

$$\text{If} \quad |x| \approx |y|, \quad \text{then} \quad p_t(x) \approx p_t(y).$$

We denote

$$p(s,x,t,y) = p_{t-s}(y-x), \quad x, y \in \mathbb{R}^d, s < t.$$

This p is the transition density of the standard isotropic α-stable Lévy process (Y_t, P^x) in $\mathbb{R}^d$ with the Lévy measure $\nu(dz) = c|z|^{-d-\alpha}dz$, and generator $\Delta^{\alpha/2}$.

To study (3), we consider nonnegative jointly Borelian $j(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d$, and we define the norm

$$\|j\| := \left(\sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} |j(z,w)| dw \right) \vee \left(\sup_{w \in \mathbb{R}^d} \int_{\mathbb{R}^d} |j(z,w)| dz \right).$$

Lemma 5.1. *There are $\eta \in [0,1)$ and $c < \infty$ such that*

$$\int_s^t du \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw\, p(s,x,u,z) j(z,w) p(u,w,t,y) \le [\eta + c(t-s)] p(s,x,t,y), \quad (16)$$

if $\|j\| < \infty$, $|j(z,w)| \le \varepsilon |w-z|^{-d-\alpha}$ and $\varepsilon > 0$ is sufficiently small.

Proof. Denote $I = p(s,x,u,z)j(z,w)p(u,w,t,y)$. Consider three sets $A_1 = \{(z,w) \in \mathbb{R}^d \times \mathbb{R}^d : |z-y| \le 4\}$, $A_2 = \{(z,w) \in \mathbb{R}^d \times \mathbb{R}^d : |w-x| \le 4|z-x|\}$ and $B = \{(z,w) \in \mathbb{R}^d \times \mathbb{R}^d : |z-x| \le \frac{1}{3}|y-x|,\ |w-y| \le \frac{1}{3}|y-x|\}$. The union of A_1, A_2 and B gives the whole of $\mathbb{R}^d$.

If $|z-y| \le 4|w-y|$, then $p(u,w,t,y) \le c_1 p(u,z,t,y)$, and by (14),

$$\begin{aligned}
\int_s^t du \iint_{A_1} dzdw\, I &\le c_1 \int_s^t du \iint_{A_1} dzdw\, p(s,x,u,z)j(z,w)p(u,z,t,y) \\
&\le c_1 \|j\| \int_s^t du \int_{\mathbb{R}^d} dz\, p(s,x,u,z)p(u,z,t,y) \\
&= c_1 \|j\| (t-s) p(s,x,t,y),
\end{aligned}$$

which is satisfactory, see (5.1). The case of A_2 is similar. For B we first consider the case $t - s \le 2|y-x|^\alpha$, and we obtain

$$\int_s^t du \iint_B dzdw\, I \le \int_s^t du \iint_B dzdw\, p(s,x,u,z)\varepsilon|w-z|^{-d-\alpha} p(u,w,t,y),$$

$$
\begin{aligned}
&\leq\ 3^{d+\alpha}\varepsilon \int\limits_s^t du \iint\limits_B dzdw\ p(s,x,u,z)|y-x|^{-d-\alpha}p(u,w,t,y)\\
&\leq\ 3^{d+\alpha}\varepsilon \int\limits_s^t du \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} dzdw\ p(s,x,u,z)p(u,w,t,y)\\
&=\ 3^{d+\alpha}\varepsilon|y-x|^{-d-\alpha}(t-s) \approx 3^{d+\alpha}\varepsilon p(s,x,t,y).
\end{aligned}
$$

In the case $t-s > 2|y-x|^\alpha$ we obtain

$$
\begin{aligned}
\int\limits_s^t du \iint\limits_B dzdw\ I \ &=\ \int\limits_s^{\frac{s+t}{2}} du \iint\limits_B dzdw\ p(s,x,u,z)j(z,w)p(u,w,t,y)\\
&\quad + \int\limits_{\frac{s+t}{2}}^t du \iint\limits_B dzdw\ p(s,x,u,z)j(z,w)p(u,w,t,y)\\
&\leq\ \int\limits_s^{\frac{s+t}{2}} du \iint\limits_B dzdw\ p(s,x,u,z)j(z,w)(t-u)^{-\frac{d}{\alpha}}\\
&\quad + \int\limits_{\frac{s+t}{2}}^t du \iint\limits_B dzdw\ (u-s)^{-\frac{d}{\alpha}}j(z,w)p(u,w,t,y)\\
&\leq\ \int\limits_s^{\frac{s+t}{2}} du \iint\limits_B dzdw\ p(s,x,u,z)j(z,w)\left(\frac{t-s}{2}\right)^{-\frac{d}{\alpha}}\\
&\quad + \int\limits_{\frac{s+t}{2}}^t du \iint\limits_B dzdw\ \left(\frac{t-s}{2}\right)^{-\frac{d}{\alpha}} j(z,w)p(u,w,t,y)\\
&\leq\ 2^{\frac{d}{\alpha}}\|j\|(t-s)^{-\frac{d}{\alpha}}(t-s) \approx 2^{\frac{d}{\alpha}}\|j\|(t-s)p(s,x,t,y).
\end{aligned}
$$

We can take $\eta = 3^{d+\alpha}\varepsilon$ and $c = c_1\|j\| + 2^{d/\alpha}\|j\|$ in (16). □

In what follows, $\widetilde{p}$ will denote the perturbation of p by $J(s,x,dt,dy) = j(x,y)\delta_s(dt)dy$, and $\widetilde{p}^-$ will be the perturbation of p by $-J$. In view of Theorem 2.4 and (12) we obtain the following result.

Corollary 5.2. *If* (16) *holds with* $0 \leq \eta < 1$, *then for* $s,t \in \mathbb{R}$, $x,y \in \mathbb{R}^d$,

$$
\widetilde{p}(s,x,t,y) \leq p(s,x,t,y)\left(\frac{1}{1-\eta}\right)^{1+c(t-s)/\eta}, \tag{17}
$$

and

$$p(s,x,t,y)\left(\frac{1-\eta}{2}\right)^{1+2c(t-s)/(1-\eta)} \leq \widetilde{p}^{-}(s,x,t,y) \leq p(s,x,t,y).$$

If $j(z,w)=j(w,z)$, then the estimates agree with those obtained in [6].

We will verify that $\widetilde{p}$ is the fundamental solution of $\Delta^{\alpha/2}+J$, i.e.

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^d} \widetilde{p}(s,x,t,y)[\partial_t+\Delta_y^{\alpha/2}+j(x,y)]\phi(t,y)dydt = -\phi(s,x), \tag{18}$$

provided (16) holds with $0\leq\eta<1$. Here and below $s\in\mathbb{R}$, $x\in\mathbb{R}^d$, and ϕ is a smooth compactly supported function on $\mathbb{R}\times\mathbb{R}^d$. By (13) (see also [4]),

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^d} p(s,x,t,y)[\partial_t+\Delta_y^{\alpha/2}]\phi(t,y)dydt = -\phi(s,x). \tag{19}$$

We denote $P(s,x,dt,dy)=p(s,s,t,y)dtdy$, $(L\phi)(s,x)=\partial_t\phi(s,x)+\Delta_y^{\alpha/2}\phi(s,x)$ and $\widetilde{P}(s,x,dt,dy)=\widetilde{p}(s,x,t,y)dtdy$. By (19), $PL\phi=-\phi$. By (1) and (17),

$$\widetilde{P}(L+J)\phi = PL\phi+\sum_{n=1}^{\infty}(PJ)^n PL\phi+\sum_{n=0}^{\infty}(PJ)^{n+1}\phi=-\phi, \tag{20}$$

where the series converge absolutely. This proves (18). We see that the argument is quite general, and hinges only on the convergence of the series (see [5, Lemma 4 and 5] for more insight).

To illustrate the influence of J on jump intensity of Markov processes, we return to the setting of Theorem 2.3, and we consider k being the transition probability of a Lévy process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ ([10]). Let $\nu(dy)$ be the Lévy measure, i.e. the jump intensity of (X_t). We have $k(s,x,t,A)=\varrho_{t-s}(A-x)$, where ϱ_t is the distribution of X_t. Let μ be a finite measure on $\mathbb{R}^d$ and $j(s,x,dy)=\mu(dy-x)$ for all s. By induction we verify that

$$k_n(s,x,t,dy)=\frac{(t-s)^n}{n!}\varrho_{t-s}*\mu^{*n}(dy-x).$$

Therefore,

$$\widetilde{k}(s,x,t,dy)=\varrho_{t-s}*\sum_{n=0}^{\infty}\frac{(t-s)^n}{n!}\mu^{*n}(dy-x),$$

and so

$$e^{-(t-s)|\mu|}\widetilde{k}(s,x,t,dy) \tag{21}$$

is the transition probability of a Lévy process with the Lévy measure $\nu + \mu$. Thus, perturbing k by j adds jumps and some mass to (X_t), and perturbing by $-j$ reduces jumps and mass of (X_t), as long as $\nu - \mu$ is nonnegative.

We like to note that subtracting jumps may destroy our (local in time, global in space) comparability of k and $\widetilde{k}^-$. Indeed, we can make $\nu(dz)-\mu(dz)$ a compactly supported Lévy measure, whose transition probability has a different, superexponential decay in space (compare [11, Lemma 2] and (15)). We see that the smallness of ε in Lemma 5.1 is crucial for Corollary 5.2.

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